INVESTIGATION OF THE STABILITY OF A STATIONARY SOLUTION OF A SYSTEM OF EQUATIONS FOR THE PLANE MOVEMENT OF AN INCOMPRESSIBLE VISCOUS LIQUID

(ISSLEDOVANIE USTOICHIVOSTI STATSIONARNOGO RESHENIIA ODNOI SISTEMY URAVNENII PLOSKOGO DVIZHENIIA NESZHIMAEMOI VIAZKOI ZHIDKOSTI)

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It is well known that, even in the simplest cases, the study of the stability of the laminar flow of an incompressible viscous liquid entails great mathematical difficulties. However, there is one particular case of hydrodynamic equations in which, under somewhat idealized conditions for the flow of liquid, such a study may be carried out to the end without great difficulty.

1. Let us examine a system of equations for the plane movement of an incompressible viscous liquid in a plane xy

$$\begin{gathered}\frac{\partial u}{\partial t}+u_x u+u_y v=-\frac{\partial p}{\partial x}+F_1+\nu\triangle u\\ \frac{\partial v}{\partial t}+v_x u+v_y v=-\frac{\partial p}{\partial y}+F_2+\nu\triangle v\\ u_x+v_y=0\end{gathered}\tag{1.1}$$

Here, u and v are the velocity projections on the x- and y-axes; the density is everywhere considered as equal to one; it is assumed that an external force F per unit mass is acting on the liquid, its components along the x- and y-axes being, respectively

$$F_1=\gamma\sin y,\qquad F_2=0\qquad(\gamma>0)$$

The idealization of conditions will consist in that, instead of the no-slip boundary conditions which are usually studied, we will be examining solutions of system (1.1) in a class of functions with a period of 2π along y. The system (1.1) has the stationary solution

$$u=\frac{\gamma}{\nu}\sin y,\qquad v=0,\qquad p=\text{const}\tag{1.2}$$

The velocity curve given by this solution contains an inflexion point. It is therefore natural to expect that at high Reynolds' numbers this flow will be unstable. The task of examining the stability of the laminar solution (1.2) of system (1.1) was placed before us by A.N. Kolmogorov at a seminar conducted by him [1].

We study the stability of solution (1.2) by the method of small disturbances. The assumption

$$\int_{-\pi}^{\pi} u(x, y)\,dy = 0$$

which assures the absence of any systematic displacement, permits us to introduce a stream function which will be periodic along y.

The stream function of infinitely small disturbances $\phi = \phi(x, y, t)$ satisfies the equation (see, for example, [2], Sections 1,2)

$$\frac{\partial}{\partial t}\Delta\varphi + \frac{\gamma}{\nu}\sin y\,\frac{\partial}{\partial x}(\varphi + \Delta\varphi) = \nu\Delta^2\varphi \tag{1.3}$$

Δ is a Laplace operator. The periodicity of ϕ along y permits us to use Fourier series. Let us seek a solution ϕ of the form

$$\varphi(x, y, t) = e^{\sigma t}\sum_{-\infty}^{\infty} c_n e^{i(\alpha x + ny)}$$

We then obtain from (1.3), for coefficients c_n, the system of equations

$$\frac{2\nu}{\gamma\alpha}(\alpha^2 + n^2)\left[\nu(\alpha^2 + n^2) + \sigma\right]c_n + c_{n-1}\left[\alpha^2 - 1 + (n-1)^2\right] - c_{n+1}\left[\alpha^2 - 1 + (n+1)^2\right] = 0 \tag{1.4}$$

In this paper we examine the sign of the real part of those values of σ for which there is a nontrivial solution of system (1.4) which tends toward zero when $|n| \to \infty$. From these further results we may draw the following conclusions.

1) When $\alpha > 1$, the real part of σ is always negative; i.e. the solution (1.2) is stable.

2) The values of σ which have a non-negative real part must be real. This also confirms the usually proposed principle of stability change (see, for example, [2], Section 2,1, p. 27).

3) It is evident from the graph of the neutral curve that when the Reynolds' number increases, instability sets in at small values of α.

2. Let us derive the equations for σ. We introduce the following notation:

$$a_n = a_n(\nu, \sigma) = \frac{2\nu}{\gamma} \frac{(\alpha^2 + n^2)[\nu(\alpha^2 + n^2) + \sigma]}{\alpha(\alpha^2 - 1 + n^2)}$$
$$d_n = d_n(\nu, \sigma) = c_n(\alpha^2 - 1 + n^2)$$

System (1.4) may now be rewritten

$$a_n d_n + d_{n-1} - d_{n+1} = 0 \tag{2.1}$$

Let us assume that system (2.1) has a solution $\{d_n\}$ which satisfies the requirements we have set. It is easily seen that in this case d_k cannot become zero for any value of k. Indeed, if $d_k = 0$ and $k > 0$, then $d_k{}' \neq 0$ when $k' = k$; otherwise the solution would be trivial. Consequently, having set $\rho_i = d_i/d_{i-1}$, where $i > k + 1$, we obtain from (2.1)

$$a_n + \frac{1}{\rho_n} = \rho_{n+1} \quad \text{при } n > k+1 \tag{2.2}$$

The solution of system (2.2) may be written in the form

$$\rho_n = a_{n-1} + \cfrac{1}{a_{n-2} + \cfrac{1}{\cdots + \cfrac{\vdots}{+ \cfrac{1}{a_{k+2} + \cfrac{1}{a_{k+1} + \cfrac{1}{\rho_{k+1}}}}}}}$$

Let us assume that σ is real and positive. Under these conditions, $a_n > 0$ when $n > 0$ and $\rho_n > a_{n-1} \to \infty$, which is impossible. If σ is complex, then we may consider system (2.1) separately for the real and imaginary parts of the coefficients a_n. The same considerations applied to the real part of a_n lead to the required result. The case $d_k = 0$, where $k < 0$, may be considered analogously.

Thus, for arbitrary values of n we may set up

$$\rho_n = \rho_n(\nu, \sigma) = \frac{d_n}{d_{n-1}} \qquad (n > 0), \qquad \mathring{\rho}_n = \mathring{\rho}_n(\nu, \sigma) = \frac{d_{n-1}}{d_n} \qquad (n < 0) \tag{2.3}$$

The following assertion will be basic for the forthcoming. If

$$\rho_1 = -\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_n + \cdots}}}} \tag{2.4}$$

then $\rho_n \to 0$, when $n \to \infty$. If equality (2.4) is not satisfied, then

$|\rho_n| \to \infty$.

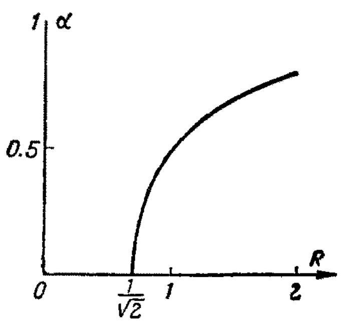
In actual fact, it follows from (2.2) that when Re $\sigma > 0$, if Re $\rho_n \geqslant 0$, then Re ρ_{n+k} is also $\geqslant 0$ (when $k > 0$), or Re $\rho_{n+k} >$ Re $a_n \to \infty (n \to \infty)$, which is impossible.

It is thus absolutely necessary that Re ρ_n be negative at any given value of n. For any stipulated σ, a number k will be found such that Re a_n will be greater than 1 for $n > k$, since Re $a_n \to \infty$, where $n \to \infty$. For $n > k$, condition Re $\rho_{n+1} < 0$, together with Equation (2.2), means that ρ_n must be placed on a complex plane, within the circle of radius 1/Re a_n, tangent to the imaginary axis and lying in the left semiplane. Repeated use of Equation (2.2) shows that ρ_{n-1} must be located within a certain circle, lying in the left semiplane and having a radius smaller than 1/Re a_n, etc. As a result, ρ_k also must lie in a certain circle in the left semiplane, having a radius smaller than 1/Re a_n. The intersection of circles constructed for ρ_k at different values of n cannot include more than one point, since the radii of these circles tend toward zero when $n \to \infty$. It is easily seen that the value ρ_k, determined by the formula

$$\rho_k = -\cfrac{1}{a_k + \cfrac{1}{a_{k+1} + \cfrac{1}{a_{k+1} + \cdots}}}$$

belongs to all these circles and, consequently, is the only possible value for ρ_k. Employing once again Equation (2.2), we easily obtain Formula (2.4). The assertion stated above is also proved by the same process.

Applying the same considerations to negative values of n, we determine that the only value of $\rho - 1$ which is reasonable for the above problem is obtained by the formula



$$\rho_{-1} = \cfrac{1}{a_{-2} + \cfrac{1}{a_{-3} + \cdots}} = \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}} \qquad (2.5)$$

From Equations (2.1) it follows, when $n = 0$, that

$$\rho_1 = a_0 + \frac{1}{a_{-1} + \rho_{-1}}$$

or, by virtue of (2.4) and (2.5)

$$-\frac{a_0}{2} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \qquad (2.6)$$

From this we conclude that in order for system (2.1) to have a solution which tends toward zero when $|n| \to \infty$, it is necessary, and also sufficient, that σ satisfy Equation (2.6).

3. An analysis of Equation (2.6) enables us to make immediately certain conclusions about stability.

The following fact pertains: if $\alpha > 1$, Equation (2.6) will have no solutions σ for which Re $\sigma > 0$.

Indeed, if $\alpha > 1$, $\mathrm{Re}(-a_0/2) < 0$. On the other hand, the expression

$$\rho_{1,k}(\nu, \sigma) = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_k}}}$$

at any value of k and with Re $\sigma \geqslant 0$ has a positive real part, and consequently under these conditions the equality (2.6) is impossible.

Refinement of this reasoning shows that when $\alpha < 1$ Equation (2.6) has only real solutions when Re $\sigma > 0$. In actual fact, let us assume for definiteness that Im $\sigma \geqslant 0$. Then arg $a_n \geqslant$ arg a_{n+1}, where $n \geqslant 0$, in which case the equality is possible only when arg $\sigma = 0$. However, with any value of k

$$\arg \rho_{1,k} \leqslant \arg a_1, \qquad \text{if} \quad \arg \rho_{1,k} < \pi$$
$$2\pi - \arg \rho_{1,k} \leqslant \arg a_1, \qquad \text{if} \quad \arg \rho_{1,k} > \pi$$

The assertion expressed above is thereby proved.

If γ and ν are changed in such a way that their relationship remains constant, the profile of the laminar flow will be unchanged. The figure depicts the approximate variation of the neutral curve for the case $2\nu/\gamma = 1$. The construction of the graph is based on equalities

$$\rho_{1,2} < \rho < \rho_{1,1} \tag{3.1}$$

which demonstrate that the graph differs from the true variation of a neutral curve when $\alpha < 1/\sqrt{2}$ by no more than 0.09 along R. It is not difficult to show that when $\alpha \to 1 - 0$, the neutral curve tends asymptotically toward a straight line $\alpha = 1$. From inequalities (3.1) we may also conclude that at small values of σ and rather large values of R, there exist positive solutions of Equation (2.6); i.e. solution (1.2) is unstable.

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